Synchronization in oscillator networks with delayed coupling: A stability criterion

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We derive a stability criterion for the synchronous state in networks of identical phase oscillators with delayed coupling. The criterion applies to any network (whether regular or random, low dimensional or high dimensional, directed or undirected) in which each oscillator receives delayed signals from k others, where k is uniform for all oscillators.

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I. INTRODUCTION

Networks of oscillators with time-delayed coupling have recently attracted attention because of their applications to neurobiology [1–3], laser arrays [4,5], microwave devices [6,7], communications satellites [8], and electronic circuits [9], and also because of their inherent mathematical interest [10–13].

In the simplest models, the oscillators are described by their phases alone, with amplitude variations neglected. For example, Schuster and Wagner considered two identical phase oscillators with delayed sinusoidal coupling [1]. Then Niebur, Schuster, and Kammen [2] studied a two-dimensional square grid of identical phase oscillators, each interacting with its four nearest neighbors, and again coupled sinusoidally with a time delay. For certain parameter values, their simulations showed that the array settles into a synchronized state in which all oscillators move in phase at a fixed frequency Ω. The stability of this in-phase state was found to depend on the values of the oscillators' natural frequency, time delay, and coupling strength. Niebur *et al.* suggested a condition for the stability of the synchronized state, based on a physical argument, but they did not address the linear stability problem mathematically. The analysis would involve studying the eigenvalues of an infinite system of linear delay-differential equations. To gain insight into this class of stability problems, Yeung and Strogatz [11] began with a simpler, idealized mean-field model in which each oscillator is coupled equally strongly to all the others, and derived a rigorous stability criterion for that special case.

Recently, we tried to extend this analysis to a one-dimensional chain of phase oscillators, each coupled to its nearest neighbors. When we did, we were surprised to find that the same stability criterion emerged as for the mean-field case. This seemed very strange—the dynamics of oscillator arrays usually depend strongly on the dimensionality of the underlying lattice, or more generally, on the topology of the array. But as we will show in this paper, for a certain class of connection topologies the condition for stable in-phase synchronization is independent of the topology. The only constraint is that each oscillator receives signals from k others, where k is uniform for all oscillators. In particular, the examples mentioned above—two oscillators, a square grid, a fully connected graph—are all included as special cases. To illustrate, we depict several connection topologies in Fig. 1 that, if populated with phase oscillators, would have identical stability criteria.

Specifically, the model is given by the following equations:

$$\dot{\theta}_i(t)=\omega+\frac{K}{k}\sum_{j=1}^{N} a_{ij} f(\theta_j(t-\tau)-\theta_i(t)), \tag{1}$$

where $\theta_i(t)$ is the phase of the ith oscillator, ω is its natural frequency, K is the coupling strength, k is the number of signals each oscillator receives, f is the coupling function, τ is the delay, and N is the total number of oscillators. The adjacency matrix a_{ij} encodes the connection topology: if oscillator j sends a signal to i, $a_{ij}=1$; otherwise, $a_{ij}=0$. This

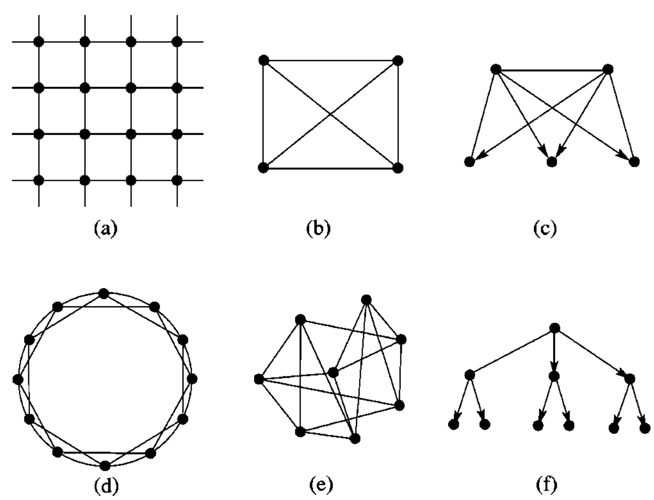

FIG. 1. Example of coupling topologies: (a) square grid with periodic boundary conditions $(k=4)$, (b) completely connected graph $(k=3)$, (c) directed graph where each oscillator receives signals from two others $(k=2)$, (d) ring with nearest neighbor and next to nearest neighbor coupling $(k=4)$, (e) randomly connected graph $(k=4)$, and (f) a tree in which the root node receives a signal from one of its children $(k=1)$. Arrows indicate direction of coupling along an edge; edges without arrows are coupled bidirectionally.

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matrix defines a directed graph in the sense that oscillators correspond to vertices, and edges correspond to coupling interactions between oscillators. Each row in the matrix sums to k. The time delay can be viewed as arising from the finite speed of signal transmission between oscillators. The normalizing prefactor $1/k$ in the coupling means that each oscillator is influenced equally by its neighbors. The in-phase synchronized state is given by

$$\theta_i(t) = \Omega t, \tag{2}$$

where the collective frequency Ω is determined implicitly by the algebraic equation

$$\Omega = \omega + K f(-\Omega\tau), \tag{3}$$

as found by direct substitution of Eq. (2) into Eq. (1). Our main result is that this state is linearly stable if and only if

$$K f'(-\Omega\tau) > 0. \tag{4}$$

II. DERIVATION OF STABILITY CONDITION

We perform a linear stability analysis to determine the local stability of solution (2) in the standard way by adding a small perturbation

$$\theta_i(t) = \Omega t + \epsilon\phi_i(t), \tag{5}$$

where $0 < \epsilon \ll 1$. To first order, the dynamics of $\phi_i(t)$ are governed by the linear delay differential equation

$$\dot{\phi}_i(t) = \frac{K}{k} f'(-\Omega\tau) \sum_{j=1}^{N} a_{ij}[\phi_j(t-\tau) - \phi_i(t)]. \tag{6}$$

If $K f'(-\Omega\tau) = 0$, we have neutral stability at linear order (in this case, higher-order terms need to be examined). From now on, and for the rest of the paper, we assume $K f'(-\Omega\tau) \neq 0$. To find an equation for the eigenvalues λ of Eq. (6), we substitute $\phi_i(t) = v_i e^{\lambda t}$ into Eq. (6) and obtain an exponential polynomial in λ:

$$k e^{\lambda\tau}[\lambda + K f'(-\Omega\tau)] v_i = K f'(-\Omega\tau) \sum_{j=1}^{N} a_{ij} v_j. \tag{7}$$

Let

$$\sigma = \frac{k e^{\lambda\tau}[\lambda + K f'(-\Omega\tau)]}{K f'(-\Omega\tau)}; \tag{8}$$

and we write Eq. (7) in matrix form

$$A v = \sigma v, \tag{9}$$

where $v = (v_1, \ldots, v_N)$. From this equation it is clear that σ is an eigenvalue of A.

Although we cannot calculate the eigenvalues without further assumptions on the topology of the graph, we can bound their locations as follows. Gerschgorin's circle theorem [14] states that every eigenvalue of a matrix $B = [b_{ij}]$ lies in at least one of the circles $C_1, \ldots, C_n$, where C_i has its center at the diagonal entry b_{ii} and its radius equal to the absolute sum along the rest of the row, i.e., the radius is equal to $\Sigma_{j\neq i}|b_{ij}|$. Applying Gerschgorin's theorem to the eigenvalues of A, we find that all the circles are the same with center at the origin, since $a_{ii} = 0$ for all i, and with radius k. Therefore all the eigenvalues of A lie within this circle, and hence satisfy

$$|\sigma| \leq k. \tag{10}$$

Now we rewrite Eq. (8) in the following form:

$$\alpha\beta e^{i\theta} = e^{\lambda\tau}(\lambda + \alpha), \tag{11}$$

where $\sigma = |\sigma| e^{i\theta}$, $\beta = (|\sigma|/k) \in [0,1]$, and $\alpha = K f'(-\Omega\tau)$.

The conditions for local stability of Eq. (2) come from the following proposition.

Proposition 1. For all λ that satisfy Eq. (11), $\mathrm{Re}(\lambda) < 0$ if and only if $\alpha > 0$.

To prove Proposition 1, let $\lambda = r + is$ and write Eq. (11) in terms of its real and imaginary parts:

$$\alpha\beta \cos(\theta - \tau s) e^{-\tau r} = r + \alpha, \tag{12}$$

$$\alpha\beta \sin(\theta - \tau s) e^{-\tau r} = s. \tag{13}$$

Squaring and adding the two equations yields

$$\alpha^2\beta^2 e^{-2\tau r} = (r+\alpha)^2 + s^2. \tag{14}$$

First, we prove the ($\Leftarrow$) direction of Proposition 1. We assume, to the contrary, that there exists a λ satisfying Eq. (11) such that $r \geq 0$, and that $\alpha > 0$. In this case $\alpha = |\alpha|$, $r = |r|$, and Eq. (14) becomes

$$\gamma = 1 + (r^2 + s^2 + 2|r||\alpha|)/\alpha^2, \tag{15}$$

where $\gamma = \beta^2 e^{-2\tau|r|}$. On the one hand, $\gamma \in [0,1]$, since $\beta \in [0,1]$ and $r \geq 0$; on the other hand, $\gamma \geq 1$, with equality only if $r = s = 0$, i.e., $\lambda = 0$. This special case corresponds to an eigenvalue $\sigma = k$, whose associated eigenvector is $(1,1,1,\ldots,1)$. This eigenvector reflects the rotational symmetry of Eq. (1); the system is neutrally stable to perturbations in which each phase is changed by the same constant amount. This is, however, the only such neutral perturbation; because the network is assumed to be connected, this eigenspace is strictly one-dimensional [14]. Hence, for all other perturbations, $\lambda \neq 0$; and therefore the right hand side of Eq. (15) is strictly greater than 1, which contradicts the earlier conclusion that $\gamma \leq 1$. Therefore, the ($\Leftarrow$) direction of the proof is complete.

Now we prove the ($\Rightarrow$) direction of Proposition 1 by proving the contrapositive. That is, we will show that if $\alpha < 0$, there exists (at least) one solution with $r \geq 0$. Since here $\alpha = -|\alpha|$, Eq. (12) becomes

$$\rho|\alpha| e^{-\tau r} = r - |\alpha|, \tag{16}$$

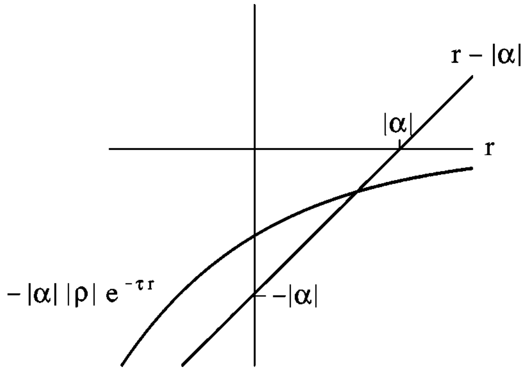

FIG. 2. Graphical solution to Eq. (18).

where $\rho = -\beta\cos(\theta - \tau s) \in [-1,1]$. First consider the case where $0 \leqslant \rho \leqslant 1$. Equation (16) can be written as

$$r = (1 + |\rho| e^{-\tau r})|\alpha|. \tag{17}$$

The right hand side of this equation is always positive, so $r > 0$ for this case; hence *all* such modes are unstable. Now consider the alternative case where $-1 \leqslant \rho < 0$. Equation (16) can be written as

$$-|\alpha||\rho| e^{-\tau r} = r - |\alpha|. \tag{18}$$

Plotting the left hand side and the right hand side of Eq. (18) versus r, we see there always exists a solution to Eq. (18) with $r > 0$, as shown by the intersection of the two curves in Fig. 2. (A negative solution also exists, but is irrelevant.) The ($\Rightarrow$) direction of the proof is now complete.

III. EXAMPLE

To illustrate our results we consider the sinusoidal coupling function $f(\theta) = \sin(\theta)$. The collective frequency Ω of the in-phase synchronous state is determined by Eq. (3) which, for this example, can be written as

$$-\frac{1}{K\tau}(\Omega\tau) + \frac{\omega}{K} = \sin(\Omega\tau). \tag{19}$$

We graphically display the solution in Fig. 3, as done in Ref. [15], by simultaneously plotting the left hand side and the right hand side of Eq. (19) versus $\Omega\tau$. The left hand side is simply a line with slope $-1/K\tau$ and horizontal intercept $\omega\tau$.

Applying the stability criterion (4) for positive K, we see that if the line intersects the positive slope of the sine curve (denoted by filled circles in Fig. 3), the in-phase synchronous state at that particular frequency Ω is stable. The state is unstable if the line intersects the negative slope of the curve, denoted by open circles.

To gain intuition on how changing parameters affects stability, first fix ω, fix τ, and increase K. This corresponds to rotating the line counterclockwise about its horizontal intercept. For very small and positive K, the line is approximately vertical and intersects the sine curve once, meaning there is one in-phase synchronous frequency Ω. Now as the horizontal intercept $\omega\tau$ is varied, the stability of the synchronous state periodically changes as the line alternates from intersecting the sine curve at a negative slope to intersecting it at a positive slope. For large K, the line is approximately horizontal and there are many intersections with the sine curve, which guarantees that there exists a stable in-phase synchronous state.

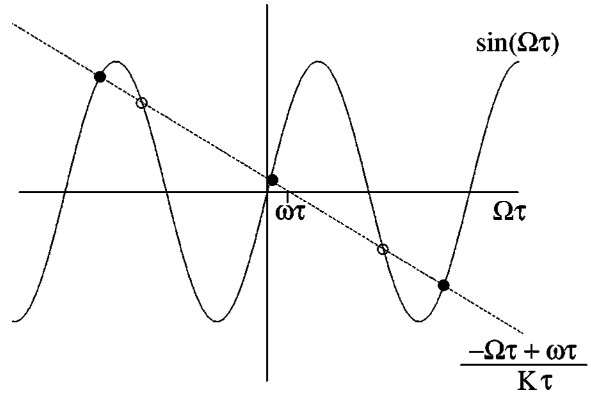

FIG. 3. Graphical solution to Eq. (19) for $K > 0$; "●" denotes a stable state and "○" denotes an unstable state.

By studying this picture further, it is clear that only the extrema of the coupling function are needed to determine stability [16]. Thus, the stability diagrams for coupling functions $f(\theta) = \sin^m(\theta)$, where $m > 0$ is odd, are identical since these functions have the same extrema. We plot this stability diagram in Fig. 4. The same diagram was found in Ref. [11] for a special case of the problem considered here. It is interesting to note that if you alter the coupling function so that the extrema with negative slope in between are closer together horizontally, the regions of instability in K/ω versus τ/T space become thinner.

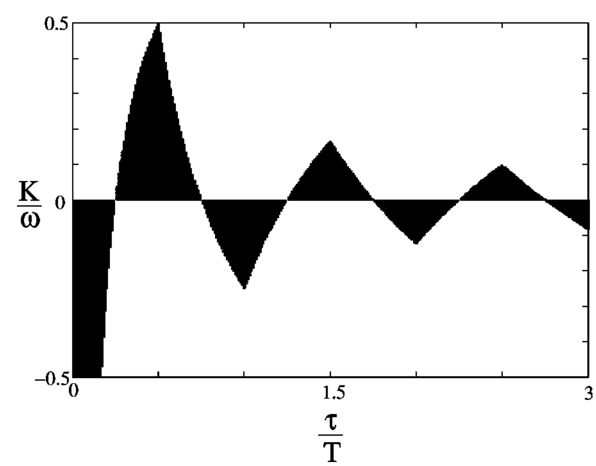

FIG. 4. Stability diagram for coupling function $f(\theta) = \sin^m(\theta)$, where $m > 0$ is odd. $T = 2\pi/\omega$ is the natural period of oscillation. In the white regions, one or more stable uniformly rotating synchronous states exist. In the shaded regions, no stable uniformly rotating synchronous states exist.

IV. DISCUSSION

Remarkably, the single condition (4) ensures that an infinite number of eigenvalues—corresponding to the infinite dimensionality of the delay-differential equation linearized about the synchronized state—is kept in the left half plane. (Generically, one would expect that an infinite number of conditions would be required.) Furthermore, the stability condition depends only on f and not on the adjacency matrix a_{ij}. In that sense, the same stability condition holds for any network in which each oscillator receives k signals, independent of all other details of its topology.

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